J. Fluid Mech. (1970), *vol.* 43, *part* 1, *pp.* 1–33
Printed in Great Britain

The non-linear refraction of shock waves by upstream disturbances in steady supersonic flow

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(Received 2 September 1969 and in revised form 12 February 1970)

The general problem studied is the propagation of an oblique shock wave through a two-dimensional, steady, non-uniform oncoming flow. A higher-order theory is developed to treat the refraction of the incident oblique shock wave by irrotational or rotational disturbances of arbitrary amplitude provided the flow is supersonic behind the shock. A unique feature of the analysis is the formulation of the flow equations on the downstream side of the shock wave. It is shown that the cumulative effect of the downstream wave interactions on the propagation of the shock wave can be accounted for exactly by a single parameter Φ, the local ratio of the pressure gradients along the Mach wave characteristic directions at the rear of the shock front. The general shock refraction problem is then reduced to a single non-linear differential equation for the local shock turning angle θ as a function of upstream conditions and an unknown wave interaction parameter Φ. To lowest order in the expansion variable $\theta\Phi$, this equation is equivalent to Whitham's (1958) approximate characteristic rule for the propagation of shock waves in non-uniform flow. While some further insight into the accuracy of Whitham's rule does emerge, the theory is not a self-contained rational approach, since some knowledge of the wave interaction parameter Φ must be assumed. Analytical and numerical solutions to the basic shock refraction relation are presented for a broad range of flows in which the principal interaction occurs with disturbances generated upstream of the shock. These solutions include the passage of a weak oblique shock wave through: a supersonic shear layer, a converging or diverging flow, a pure pressure disturbance, Prandtl–Meyer expansions of the same and opposite family, an isentropic non-simple wave region, and a constant pressure rotational flow. The comparison between analytic and numerical results is very satisfactory.

1. Introduction

In the past several decades, analytical methods have been developed for treating inviscid steady, two-dimensional, supersonic compressible motions which by and large fall into three categories: (*a*) flows with small rotationality, (*b*) flows where the disturbance field is everywhere a small departure from uniform free-stream conditions, and (*c*) flows involving the interaction of a very weakly disturbed uniform stream with a shock wave of finite strength. No equivalent analytical theory has emerged for treating supersonic flows with large amplitude irrotational or strong rotational disturbances. The present paper examines one important facet of this general problem, the non-linear refraction of oblique shock waves by incoming disturbances of arbitrary magnitude when the flow downstream of the shock is supersonic. A new shock refraction equation related to Whitham's (1958) rule for the propagation of shock waves in non-uniform flow is derived in § 3 for this purpose. This relation, a non-linear differential equation for the local turning angle of the flow at the shock, applies to the passage of an oblique shock through an arbitrary two-dimensional disturbance field. The series expansion procedure employed in the analytic solution of this differential equation differs from the usual higher-order theory in that the algebraic coefficients are expanded about the locally disturbed conditions ahead of the shock instead of an undisturbed uniform reference state. One is, thus, able to examine shock propagation through incoming flows which are large departures from uniform free-stream conditions.

In flows of type (*a*), flows with small rotationality, the mathematical approach is based on a perturbation expansion in which the associated irrotational flow is known and treated as the lowest order solution. In flows of type (*b*), both the oblique shock wave relations and the Prandtl–Meyer simple wave relation are expanded as a power series in the local flow turning angle θ (figure 1 (*a*)), about a uniform flow M_∞ at upstream infinity. This theory, which was extensively developed by numerous investigators and applied to thin airfoils, is summarized in Lighthill (1954). In flows of type (*c*), the disturbances are treated as a linear perturbation about the solution for an undisturbed shock in a uniform stream, Ribner (1954) and Chang (1957). These linear analyses require both that the incoming disturbance field be very weak and that the flow turning angle characteristic of this disturbed field be small compared to the flow turning angle for the undisturbed oblique shock. Analytical techniques for investigating the propagation of weak oblique shocks through general two-dimensional rotational and irrotational disturbances of arbitrary magnitude have not, to the authors knowledge, previously been treated in the literature and are developed in the present study.

For flows other than types (*a*), (*b*) and (*c*) above, one usually resorts to numerical procedures involving the solution of the governing system of hyperbolic partial differential equations for steady, two-dimensional, inviscid supersonic flow. These numerical procedures, which are usually based on the method of characteristics, are expensive and must be repeated for each different set of boundary data. For problems involving the propagation of shocks in non-uniform regions Whitham (1958) has proposed a simple approximate rule, for which it is not necessary to solve the full system of partial differential equations on the downstream side of the shock wave. The rule states that the flow quantities just behind the shock wave should satisfy the same differential relation as applies along the characteristic co-ordinate of the same family as the shock wave. This approximation is equivalent to neglecting wave interference effects (multiple reflections) in the disturbed region behind the shock. The rule was used by Whitham to re-derive the equations obtained by Chisnell (1955) and (1957) for the propagation of a shock wave normally through a non-uniformity in density or channel area in one-dimensional unsteady flow and the equations derived by Moeckel (1952) for the refraction of a shock wave by a parallel shear layer in two-dimensional steady supersonic flow. The original derivations of Chisnell and Moeckel were both based on an approach in which the region of non-uniformity was divided into a series of small discontinuities wherein the interaction between the shock wave and each discontinuity was treated independently of the presence of the others.

The accuracy of Whitham's simple rule has been checked against exact solutions using the method of characteristics by Bird (1961) for the one-dimensional unsteady motion of a normal shock through a density gradient and by Rosciszewski (1960) for the interaction between an oblique shock wave and a Prandtl–Meyer expansion of the opposite family in addition to other flow problems not as closely related to the present study. Considering the neglect of all wave interference effects downstream of the shock and the fact that the characteristic co-ordinate of the same family as the shock in the (x, t) or (x, y) plane departed significantly from the shock co-ordinates in some of the cases studied, the overall agreement with the exact solutions is rather surprising. This accuracy has never been satisfactorily explained and remains perhaps the most important unanswered question in the theory. The double power series expansion of the exact shock refraction relation derived in the present study provides some new insight into this basic question, although it still falls short of being a rational derivation of Whitham's rule. This ordered expansion is also used to derive an approximate non-linear equation that can be solved analytically for weak oblique shocks propagating through incident streams with large amplitude irrotational or strong rotational disturbances. Heretofore, the non-linear equations that have emerged from the application of the characteristic rule have been solved numerically except for special limiting cases.

Some aspects of the mathematical development are novel. First, the flow conditions just behind the shock are expressed as a power series expansion in the local shock turning angle θ about the local conditions ahead of the shock instead of a uniform upstream reference state as in higher-order thin airfoil theory. Thus, the coefficients in the differential relations obtained from the Rankine–Hugoniot shock jump conditions are not constants but variable functions of Mach number. A second feature is the treatment of the Euler equations on the downstream side of the shock. In simple wave regions, only a single variable, the local flow angle, is required to determine the pressure and other flow variables. The pressure coefficient can, therefore, be expanded using the Prandtl–Meyer relation as a power series in the local flow angle alone. It is this simplifying feature which enables one to determine easily the complete flow pattern to second order in two-dimensional thin airfoil theory. For strongly rotational flows, a wave of one family will necessarily generate waves of comparable strength of the opposite family. Multiple reflections become important and another variable parameter is obviously required to describe the second wave system. An important point discovered in the course of this study is that, as far as the refraction of the shock is concerned, only a single new unknown parameter, the local ratio $\Phi = p_\xi/p_\eta$ of the pressure gradients along the Mach wave characteristic directions at the rear of the shock front, is required to describe exactly the cumulative effect on the shock of all the reflected wave interactions that occur in the disturbed flow downstream. Since the η characteristic makes an angle which is of order θ with respect to the incident shock of the same family (see figure 1(*b*)), the relative strength of incident and emitted waves at the rear of the shock is not $O(\Phi)$ but $O(\theta\Phi)$. Thus, one anticipates that the expansion of the flow variables in the Euler equations applied at the rear of the shock front involves a power series in two quantities θ and $\theta\Phi$.

The outline of this paper is as follows: §2 outlines the simplifying features of the steady state refraction theory and motivates the derivation of the basic refraction relation presented in §3. The boundary conditions upstream of the shock wave are described in §4. Solutions are then presented in §5 for seven different types of incoming flows: pure shear, pure convergence and divergence, pure pressure disturbance, simple waves of the same and opposite family as the shock, irrotational non-simple wave regions, and constant pressure rotational flows. Section 6 discusses these results and compares analytical and numerical solutions and §7 is the conclusion.

2. Simplifying features

The most important simplifying feature of the shock propagation problems treated in this investigation is that principal interaction occurs with disturbances generated in the flow ahead of the shock rather than those generated in the flow behind it. One can deduce, as we do below, that for disturbances of this nature the strength of the incident waves striking the rear of the shock will be weak compared to the emitted waves produced by the interaction at the shock front even if large amplitude disturbances are present in the incident stream. One way of arriving at this conclusion is to decompose the incoming disturbances into three basic components: (1) waves of the same family as the shock, (2) waves of the opposite family to the shock, and (3) a parallel shear or vortical flow.

For waves of type (1), it is well known, from the results of higher-order thin airfoil theory, that to $O(\theta^2)$ the disturbance system is transmitted along the shock wave, and that the emitted waves generated in the interaction at the shock front due to the vorticity or entropy gradients produced by the variation in shock strength are $O(\theta^3)$. Downstream of the shock, the emitted waves will undergo a sequence of multiple reflections with the entropy layer that is convected downstream of the shock front along streamlines. However, the strength of each reflected wave will be $O(\theta^3)$ smaller than its predecessor. This follows from the entropy reflection principle, which states that the difference in p or θ between adjacent points along a wave is influenced by the entropy differences to an amount of the order of this entropy difference times the difference in p or θ. Thus, the strength of the incident wave family at the rear of the shock is $O(\theta^6)$, the primary emitted waves and the entropy layer each being $O(\theta^3)$.

For incoming disturbances of type (2), waves of the opposite family to the shock, the disturbance system is primarily transmitted through the shock front.

The variation in strength of the incident shock wave is of the order of the product of the strengths of the incident shock and the upstream disturbance wave system. If both are $O(\theta)$, then it follows from previous arguments that the attenuation of the incident shock is $O(\theta^2)$. Thus, the entropy gradients produced by the interaction at the shock front are $O(\theta^6)$, and the downstream reflections of the transmitted waves by this entropy layer are $O(\theta^7)$.

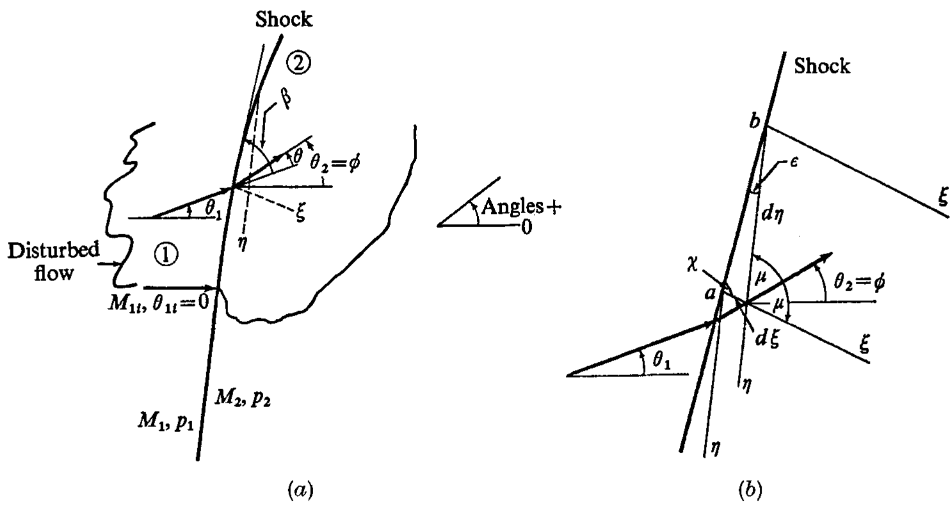

FIGURE 1. (a) Schematic of flow angles at the shock front. (b) Characteristic co-ordinates behind shock wave.

For incoming disturbances of type (3), where the vorticity distribution is produced upstream of the shock in high shear regions, and is only slightly modified by the interaction with the incident shock, the problem of multiple downstream reflections is more subtle. The ratio $\Phi = p_\xi/p_\eta$ is a measure of the local relative strength of waves of the same and opposite family downstream of the shock. Since the entropy gradients generated by a shock whose turning angle is $O(\theta)$ are at most $O(\theta^3)$, the incoming vorticity distribution is only altered to $O(\theta^3)$ by the presence of the shock wave. Φ at a given point is proportional to the ratio of the product of the cumulative strength of the waves of the opposite family that are either transmitted or emitted at the shock front below that point and the strength of the entropy layer through which they pass to the product of the strength of the incident shock and the local entropy gradient which it sees. Thus, Φ can be $O(1)$, if there is an incoming wave system of the opposite family and of the same strength as the shock, or if the attenuation of the incident shock by the vortical layer is of the same order as its undisturbed strength. In the latter case, the total strength of the emitted waves produced by the vorticity interaction at the shock front will be of the same order as the shock itself. In problems of this nature, the reason for anticipating, that incident waves will be less important than emitted waves along the rear of the shock, is the difference in projected area that a differential element of the shock makes with the ξ and η characteristics. The angle ϵ, which the shock makes with the η characteristic, is of the order of the local shock turning angle θ, whereas the angle which the shock makes with the ξ characteristics, $2\mu-\epsilon$, is to lowest order 2μ. Thus, from figure 1(*b*) the width of the η characteristic tube subtended by the shock segment (*ab*) will be $O(\theta)$ smaller than the width of the ξ characteristic tube, and the ratio of incident to emitted waves from (*ab*) will be $O(\theta\Phi)$. Consequently, incident waves striking (*ab*) due to forward scattering will be $O(\theta)$ smaller than the waves emitted from (*ab*), even when $\Phi = O(1)$. Φ cannot be $> O(1)$, if the principal interaction occurs with disturbances generated upstream of the shock.†

The above arguments show qualitatively that, for an arbitrary upstream disturbance, which contains components of types (1), (2) and (3), multiple reflections and incident waves at the rear of the shock are higher-order effects in the treatment at the rear of the shock of the wave system that carries the disturbance downstream. Therefore, one anticipates that, for this class of shock refraction problems, Whitham's approximate characteristic rule should provide a reasonable description, and that the criterion for its validity is simply that $\theta\Phi \ll 1$. The problem of the interaction of a shock with an upstream generated disturbance is, in a sense, the inverse of the leading-edge shock problems treated by Friedrichs (1948) and Whitham (1952) for the flow past a thin airfoil or projectile in a uniform stream. In the latter problems, the principal interaction occurs with incident waves emanating at the obstacle striking the rear of the shock, where their entropy reflections of the opposite family (the primary family in the present study) at the shock front are third-order effects.

An important simplifying feature in the description of the flow behind the shock is the fact that the $(dp, d\phi)$ characteristic relations do not depend on the entropy s, or stagnation enthalpy H, explicitly, and are therefore the same for both irrotational and rotational flow. For points just behind the shock front, this simplification is not just formal, as is the case for interior points in the flow field. The determination of the velocity components and Mach number need not include the other characteristic relations, which do depend on ds and dH explicitly; they can be obtained directly from the local oblique shock relations. The vorticity layer on the downstream side of the shock wave can be thought of as a continuous distribution of slip lines. The matching condition, which relates the oblique shock relations and the Euler equations at the rear of the shock, is, therefore, basically a slip-line boundary condition between the local pressure and flow deflection angle. Thus, both the flow equations and boundary conditions motivate the choice of the pressure behind the shock p_2, and the flow turning angle θ, as the natural dependent variables for shock refraction problems.

These basic features are employed in § 3 to derive a new shock refraction relation for rotational steady flow. The key steps to be followed in this derivation are:

(1) The oblique shock relations are differentiated along the shock front. All the unknown differentials on the downstream side are eliminated in favour of

† Once the shock has passed through the region where the interaction with upstream disturbances dominates, Φ will become large, since the reflected waves incident at the rear of the shock then become the principal wave system. These waves interact with the shock over large distances, but will be important only if their cumulative strength is comparable to that of the shock itself.

dp_2 and $d\theta$. The result is a differential equation relating dp_2 and $d\theta$ to the differential changes dp_1 and dM_1 on the upstream side of the shock (see (3.5)).

(2) The coefficients of the differential equation derived in (1) are developed as a power series in θ about the local conditions upstream of the shock wave (see (3.8)).

(3) A second independent relation between dp_2 and $d\theta$ at the rear of the shock is derived from the $(dp, d\phi)$ characteristic relations on the downstream side of the shock front (see (3.23)).

(4) The coefficients of the differential equation derived in (3) depend on the ratio, $\Phi = p_\xi/p_\eta$, and the flow variables, M_2, p_2, etc., on the downstream side of the shock. These coefficients are developed as a power series in θ and $\theta\Phi$ about the local conditions ahead of the shock wave (see (3.24)).

(5) The slip line compatibility or matching condition on p_2 and θ at the rear of the shock is now satisfied by equating the two relations for dp_2 and $d\theta$. The result is a single equation relating the differential changes in shock turning angle $d\theta$ to the differential changes dp_1, $d\theta_1$, dM_1 of the flow variables on the upstream side of the shock (see (3.28)).

Equation (3.28), the new basic shock refraction relation, describes the passage of an oblique shock through an arbitrary disturbed flow for which $M_2 > 1$.

3. Derivation of shock refraction relation

(i) *Oblique shock wave equations*

First, we derive the results numbered (1) and (2) in § 2. The oblique shock relations apply locally, and they are valid across the shock throughout the disturbed flow region. The basic unknown is the local flow turning angle at the shock, $\theta = \theta_2 - \theta_1$. The subscripts 1 and 2 refer to conditions just upstream and downstream of the shock respectively. The local pressure jump across the shock is given by

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 \sin^2\beta - 1), \tag{3.1}$$

where β, the local shock wave angle, is related to θ by

$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{(\gamma-1)M_1^2\sin^2\beta + 2}{(\gamma+1)M_1^2\sin^2\beta}. \tag{3.2}$$

Equations (3.1) and (3.2) provide two equations for the three unknowns p_2, β and θ. Taking the differentials of (3.1) and (3.2), one obtains, respectively,

$$dp_2 - \left(\frac{2\gamma}{\gamma+1}\right) p_1 M_1^2 \sin 2\beta\, d\beta = \left(\frac{4\gamma}{\gamma+1}\right) p_1 M_1 \sin^2\beta\, dM_1 + \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2\sin^2\beta - 1)\right] dp_1, \tag{3.3}$$

$$-d\theta + \left(1 - \frac{\sin 2(\beta-\theta)}{\sin 2\beta} + \frac{4}{\gamma+1}\frac{\cos^2(\beta-\theta)}{M_1^2\sin^2\beta}\right) d\beta = -\left(\frac{8}{\gamma+1}\right)\frac{\cos^2(\beta-\theta)}{\sin 2\beta}\frac{dM_1}{M_1^3}. \tag{3.4}$$

$d\beta$ can now be eliminated between (3.3) and (3.4) and the first result obtained:

$$\left.\begin{aligned} dp_2 - ap_1 d\theta &= b\,dp_1 + acp_1 dM_1, \\ a &= \frac{2\gamma}{\gamma+1}\frac{M_1^2 \sin 2\beta}{\alpha}, \\ b &= 1 + \left(\frac{2\gamma}{\gamma+1}\right)(M_1^2 \sin^2\beta - 1), \\ c &= \frac{\tan\beta}{M_1}\left(1 - \frac{\sin 2(\beta-\theta)}{\sin 2\beta}\right), \\ \alpha &= 1 - \frac{\sin 2(\beta-\theta)}{\sin 2\beta} + \left(\frac{4}{\gamma+1}\right)\frac{\cos^2(\beta-\theta)}{M_1^2 \sin^2\beta}. \end{aligned}\right\} \tag{3.5}$$

Equation (3.5) relates the differential changes in pressure along the rear of the shock dp_2 and the differential changes in the local flow turning angle at the shock $d\theta$ to the incoming disturbances dp_1 and dM_1. The equation is non-linear, since the coefficients a, b, c depend on the unknowns β and θ, where $d\beta$ from (3.4) is given by

$$\left.\begin{aligned} d\beta - \frac{1}{\alpha}\,d\theta &= e\,dM_1, \\ e &= -\frac{8}{\gamma+1}\left(\frac{1}{\alpha M_1^3}\right)\frac{\cos^2(\beta-\theta)}{\sin 2\beta}. \end{aligned}\right\} \tag{3.6}$$

From (3.6), $\beta = \beta(\theta, M_1)$. Thus, the coefficients a, b, c in (3.5) are functions of θ and M_1 only. For a prescribed distribution of M_1 and p_1 ahead of the shock wave, the right-hand sides of (3.5) and (3.6) are known, and (3.5) provides a single differential equation for the two unknowns p_2 and θ. No additional information about p_2 and θ or θ_2 can be obtained from the oblique shock relations. A second independent relationship between dp_2 and $d\theta$ or $d\theta_2$ is, therefore, required, and will be derived in §3(ii) from the equations governing the flow downstream of the shock.

The flow variable θ_1 does not appear explicitly in (3.5) and (3.6). The relevant angle for the oblique shock wave is the local flow turning angle θ across the shock. Therefore, one anticipates that for weak shocks the coefficients a, b, c, e, α may be expressed as power series in θ about the *local* conditions that obtain upstream of the shock. This expansion can be performed in the same manner as the usual higher-order theory (see Lighthill 1954). The only difference is that the expansion uses the local Mach number M_1, which is variable, as the upstream reference condition rather than the constant M_∞, as would apply for a uniform upstream state. One obtains for $\tan\beta$

$$\tan\beta(M_1, \theta) = \frac{1}{\omega_1} + \frac{\gamma+1}{4}\frac{M_1^4}{\omega_1^4}\theta + O(\theta^2), \tag{3.7}$$

where $$\omega_1 = (M_1^2 - 1)^{\frac{1}{2}}$$

(Lighthill 1954). The trigonometric terms in the coefficients a, b, etc. in (3.5) and (3.6) can be expressed in terms of $\tan\beta$ and trigonometric functions of θ.

Expanding the latter as a power series in θ, and using (3.7), one obtains result 2 of §2:

$$\left.\begin{aligned} a &= a_1 + a_2\theta + O(\theta^2),\\ b &= b_1 + b_2\theta + O(\theta^2),\\ c &= c_1 + c_2\theta + c_3\theta^2 + O(\theta^3),\\ e &= e_1 + e_2\theta + e_3\theta^2 + O(\theta^3),\\ \alpha &= \alpha_1 + \alpha_2\theta + O(\theta^2), \end{aligned}\right\} \tag{3.8}$$

$$a_1 = \frac{\gamma M_1^2}{\omega_1}, \quad a_2 = \frac{\gamma M_1^2(M_1^2-2)^2 + \gamma^2 M_1^6}{2\omega_1^4},$$

$$b_1 = 1, \quad b_2 = a_1,$$

$$c_1 = 0, \quad c_2 = \frac{M_1^2 - 2}{M_1\omega_1^2}, \quad c_3 = \frac{2}{M_1\omega_1}\left(1 - \frac{\gamma+1}{4}\frac{M_1^4}{\omega_1^4}\right),$$

$$\alpha_1 = \frac{4}{\gamma+1}\left(\frac{\omega_1}{M_1}\right)^2, \quad \alpha_2 = \frac{M_1^2-4}{\omega_1} + \frac{8}{\gamma+1}\left(\frac{\omega_1}{M_1^2}\right),$$

$$e_1 = -\frac{1}{M_1\omega_1}, \quad e_2 = -\frac{1}{\alpha_1^2}\frac{d\alpha_1}{dM_1}, \quad e_3 = -\frac{1}{2\alpha_1}\left(\frac{d\alpha_2}{dM_1} - \frac{\alpha_2}{\alpha_1}\frac{d\alpha_1}{dM_1}\right).$$

The a_j, b_j, etc., are, therefore, variable functions of M_1 alone. Terms of $O(\theta^2)$ are retained in the expansion for c, since $c_1 = 0$, and the first higher-order correction for the pure shear case will require c_3. The term $e_3\theta^2$ is retained, because it is required in the second-order correction for the shock wave angle β. To $O(\theta^2)$, equation (3.6) is

$$d\beta = \left(\frac{1}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\theta\right)d\theta + (e_1 + e_2\theta + e_3\theta^2)\,dM_1. \tag{3.9}$$

However, $e_1 dM_1 = d\mu_1$, where μ_1 is the local Mach angle, $\sin^{-1}(1/M_1)$, of the incident stream. Combining this with the expressions for the e_j from (3.8), one notes that the right-hand side of (3.9) is the exact differential,

$$d\beta = d\mu_1 + d\left(\frac{\theta}{\alpha_1}\right) - \tfrac{1}{2}d\left(\frac{\alpha_2}{\alpha_1}\theta^2\right). \tag{3.10}$$

Thus, to $O(\theta^2)$

$$\beta - \beta_i = \mu - \mu_i + \frac{\theta}{\alpha_1} - \frac{\theta_i}{\alpha_{1i}} - \frac{1}{2}\left(\frac{\alpha_2}{\alpha_1}\theta^2 - \frac{\alpha_{2i}}{\alpha_{1i}}\theta_i^2\right), \tag{3.11}$$

where the subscript i refers to any convenient reference state, e.g. the conditions at the point where the shock first enters the disturbed flow region.

For a uniform incident stream, (3.5) for weak shocks reduces to

$$dp_2 - [a_1 + a_2\theta + O(\theta^2)]\,p_1 d\theta = 0. \tag{3.12}$$

The integral of (3.12),

$$\frac{p_2 - p_1}{p_1} = a_1\theta + \frac{a_2}{2}\theta^2 + O(\theta^3), \tag{3.13}$$

is Busemann's well-known second-order expression for the wave strength.

(ii) *Flow equations behind the shock wave*

We next derive the second independent equation relating dp_2 and $d\theta$ at the rear of the shock. The important simplifying feature to be used is that the (p, ϕ) form of the characteristic relations,

$$dp \pm m\,d\phi = 0 \quad \text{on} \quad \frac{dy}{dx} = \tan(\phi \pm \mu), \tag{3.14}$$

where

$$m = \gamma p M^2 \tan\mu$$

(Ferri 1954, p. 590, (2.25)) is the same for both irrotational and rotational flow, since the differential variations in entropy and stagnation enthalpy do not appear explicitly in this form of the relationships. At the rear of the shock wave, $p = p_2$ and $\phi = \theta_2$. The principal complication is that the (p, ϕ) relationship is required along a line at the rear of the shock, where the Euler equations do not have the simple ordinary differential form of (3.14). First, we shall write the equation of the rear of the shock front $Y(x)$ in (ξ, η) characteristic co-ordinates, and then derive the relation between dp and $d\phi$ that exists along the curve $Y(x)$.

In figure 1 (*b*), ϵ and χ are the two interior angles that the η and ξ characteristics, respectively, make with the rear of the shock wave front. Since the sum of the interior angles of a triangle sum to 180 degrees,

$$\chi = 2\mu - \epsilon. \tag{3.15}$$

Thus, along the shock front, $d\xi$ and $d\eta$ are related by

$$d\eta \sin \epsilon = d\xi \sin (\pi - \chi). \tag{3.16}$$

Combining these two results, one has

$$d\xi = \sigma \, d\eta, \tag{3.17}$$

$$\sigma = \frac{\sin \epsilon}{\sin (2\mu_2 - \epsilon)}.$$

Along the curve $Y(x)$ at the rear of the shock front defined by (3.17), $dp = dp_2$ and $d\phi = d\theta_2$. Therefore,

$$\left.\begin{aligned} dp_2 &= (p_\eta + \sigma p_\xi)\, d\eta, \\ d\theta_2 &= (\phi_\eta + \sigma \phi_\xi)\, d\eta. \end{aligned}\right\} \quad \begin{aligned}(3.18)\\(3.19)\end{aligned}$$

Dividing (3.18) by (3.19), and rearranging terms,

$$dp_2 = \frac{p_\eta}{\phi_\eta}\left(\frac{1+\sigma(p_\xi/p_\eta)}{1+\sigma(\phi_\xi/\phi_\eta)}\right) d\theta_2 \quad \text{at} \quad Y(x). \tag{3.20}$$

However, p_ξ/p_η and ϕ_ξ/ϕ_η are related through the characteristic relations (3.14):

$$\frac{p_\xi}{\phi_\xi} = -\frac{p_\eta}{\sigma_\eta}. \tag{3.21}$$

Substituting this last result back in (3.20), and noting from (3.14) that

$$\frac{p_\eta}{\phi_\eta} = -m_2$$

at the rear of the shock, one obtains

$$dp_2 + m_2\left(\frac{1+\sigma\Phi}{1-\sigma\Phi}\right) d\theta_2 = 0 \quad \text{at} \quad Y(x), \tag{3.22}$$

where $\Phi = p_\xi/p_\eta$, the ratio of the pressure gradients along the ξ and η characteristics, is a measure of the relative strength of the incident and emitted waves behind the shock. The interesting feature of (3.22) is that the entire wave interference effect downstream of the shock is described by a single unknown parameter Φ. Equation (3.22) reduces to Whitham's characteristic rule, if the factor

$$(1+\sigma\Phi)/(1-\sigma\Phi)$$

is set equal to unity.

The factor $(1+\sigma\Phi)/(1-\sigma\Phi)$ in (3.22) can be written as $1+\{2\sigma\Phi/(1-\sigma\Phi)\}$, and (3.22) rearranged to read

$$dp_2+m_2 d\theta_2 = -\left(\frac{2\sigma\Phi}{1-\sigma\Phi}\right) m_2 d\theta_2. \tag{3.22a}$$

This form is convenient for examining the accuracy of Whitham's approximation, since from (3.14) the left-hand side of (3.22a) is zero along the characteristic of the same family as the shock. In Whitham's approximate rule, the left-hand side of (3.22a) is set equal to zero and applied along the shock front. Thus, the deviation of $dp_2+m_2 d\theta_2$ from zero along the shock is one measure of the accuracy of Whitham's simple prescription. Just why this group of terms should be small is difficult to explain, but comparisons of exact solutions with solutions using the rule have yielded surprisingly good agreement. Some insight into this basic question can be had by examining the right-hand side of (3.22a). It is evident that for Whitham's rule to be accurate either $\sigma\Phi$ or $d\theta_2$ must be small. Since, as we shall show later, $d\theta_2 = 0$ to lowest order only if the upstream disturbance is comprised solely of simple waves of the same family as the shock, one concludes that the accuracy of Whitham's rule depends on the smallness of $\sigma\Phi$. For weak shocks, from (3.17), σ is proportional to the shock turning angle θ, since ϵ can be developed as a power series in θ. The magnitude of the geometrical factor σ is computed numerically for both weak and stronger oblique shocks later in §3 (ii) (see figure 2). σ increases from zero for a Mach wave to a maximum value of unity when $M_2 = 1$. Thus, for very weak shocks, the accuracy of the characteristic rule follows from the closeness in slope of the η characteristic surface and the shockwave. For stronger oblique shocks, the accuracy of the rule must rest on the fact that the interaction parameter $\Phi \ll 1$. Based on reasons discussed in §2, Φ is always $< O(1)$ in problems where the principal interaction occurs with disturbances generated upstream of the shock. However, for other applications, Φ need not be small compared to unity, and the accuracy of the approximate rule must be carefully scrutinized unless $\sigma \ll 1$.

The unexpected accuracy of the results obtained by Chisnell and Whitham for one-dimensional unsteady shock propagation using the approximate rule was questioned in Whitham (1958) using arguments similar to those presented above, except that characteristic co-ordinates are not introduced. We shall develop his arguments for two-dimensional steady flow here, and show the equivalance of the two approaches for one-dimensional unsteady flow in §7. Along the shock, the right-hand side of (3.22a) can be expressed in (x, y) co-ordinates as

$$dp_2+m_2 d\theta_2 = (\tan(\theta_1+\beta))(p_y+m_2\theta_y)\,dx+(p_x+m_2\theta_x)\,dx,$$

where $\tan(\theta_1+\beta)$ is the local slope of the shock wave. Similarly, along the η characteristic, one can write

$$p_x+(\tan(\theta_2+\mu_2))\,p_y+m_2(\theta_x+(\tan(\theta_2+\mu_2))\,\theta_y) = 0.$$

Combining these last two results, one can show that, along the shock wave,

$$dp_2+m_2 d\theta_2 = (\tan(\theta_1+\beta)-\tan(\theta_2+\mu_2))(p_y+m_2\theta_y)\,dx. \qquad (3.22b)$$

The right-hand sides of (3.22a) and (3.22b) are equal. The first factor in (3.22b), the difference in slope between the local shock wave and η characteristic directions, is related to our σ. For stronger oblique shocks, this geometrical factor can be of $O(1)$, and the accuracy of the approximate rule depends on the smallness of the second factor $p_y+m_2\theta_y$. From (3.14) and (3.21), this second factor can be written in characteristic co-ordinates as

$$p_y+m_2\theta_y = 2p_\xi \xi_y,$$

and hence is related to our Φ. Since it is the relative magnitude of p_ξ and p_η, and not the absolute value of p_ξ, that allows one to neglect wave interference effects, Φ for present purposes is the more suitable parameter.

Since θ, and not θ_2, is the basic variable for the oblique shock relation (3.5), it is convenient to rewrite (3.22) as

$$dp_2+h\,d\theta = -h\,d\theta_1, \qquad (3.23)$$

$$h = m_2\left(\frac{1+\sigma\Phi}{1-\sigma\Phi}\right).$$

Equation (3.23) provides the second independent relation between dp_2 and $d\theta$. This is result number (3) of § 2. It will be used with the other independent relation (3.5) between dp_2 and $d\theta$ derived in § 3 (i). Equation (3.23) involves no approximations other than those inherent in the Euler equations.

For the reasons discussed in § 2, Φ will be $\leqslant O(1)$ in regions where the interaction between the shock and the incident stream is the dominant effect. Both σ and the flow variables with the subscript 2 in the expression for h can be developed as a power series in θ about the local conditions ahead of the shock. To determine the leading terms in the series for σ, one uses the knowledge that, to $O(\theta)$, the shock bisects the η characteristics ahead of and behind itself. Therefore, to $O(\theta)$, $\epsilon = (\mu_2-\mu_1)/2$. The power series for p_2, M_2 and μ_2 are readily obtained from the oblique shock relations. Consequently, the series expansion for h takes the form of a double power series in θ and $\theta\Phi$:

$$h = h_1+h_2\theta+h_3\theta\Phi+O(\theta^2)+O(\theta^2\Phi^2), \qquad (3.24)$$

$$h_1 = a_1 p_1, \quad h_2 = a_2 p_1, \quad h_3 = a_1 p_1 g_1,$$

where the a_j are given in (3.8),

$$g_1 = \frac{\gamma+1}{4}\left(\frac{M_1^4}{\omega_1^3}\right),$$

and $\sigma = \frac{1}{2}g_1\theta+O(\theta^2)$. Equation (3.24) is result (4) of § 2. While h can be formally expanded in the double power series (3.24), this cannot be viewed as a rational approach to Whitham's rule. Φ is unknown, and some knowledge of its order of magnitude must be assumed if one is to neglect certain higher-order terms in the series.

Significant simplification is obtained if the third term in the above expansion for h can be neglected in comparison with the second. The second term in the expansion for h (3.24) will be greater than the third if

$$\frac{a_1 g_1}{a_2}\Phi < 1. \tag{3.25}$$

The coefficient of Φ in (3.25) is

$$\frac{a_1 g_1}{a_2} = \frac{(\gamma+1)\,M_1^4}{2[(\gamma+1)\,M_1^4 - 4(M_1^2-1)]}.$$

For a $\gamma = 1{\cdot}4$ gas, this coefficient $a_1 g_1/a_2$ increases from the value $\frac{1}{2}$ at $M_1 = 1$ to a maximum value $\frac{6}{7}$ at $M_1 = \surd 2$ and then decreases back toward $\frac{1}{2}$ as $M_1 \to \infty$. Thus, $h_2 > h_3$ over the entire range of M_1, and the minimum value of Φ, for which the third term in (3.24) can exceed the second term, is $\frac{7}{6}$ to 2, depending on M_1. The solutions examined in the present investigation will assume that condition (3.25) is satisfied. For weak oblique shocks, this condition ensures that incident waves at the rear of the shock will not have an important effect to $O(\theta^3)$ for upstream shear disturbances, and to $O(\theta^2)$ for upstream wave type disturbances. (See (3.30).)

For oblique shocks with larger turning angles, the equivalent criterion to (3.25), for the neglect of multiple wave reflections behind the shock in the expression for h in (3.23), is that

$$\sigma\Phi = \frac{\sin\epsilon}{\sin(2\mu-\epsilon)}\,\Phi \ll 1. \tag{3.26}$$

If this condition is satisfied, h then reduces to the value it takes in Whitham's approximate characteristic rule:

$$h = m_2. \tag{3.27}$$

The quantity σ depends only on the local oblique shock wave relations. Referring to figures 1 (*a*) and (*b*) for the geometric relationship between the various angles, one can show, after some trigonometric manipulation, that the expression (3.17) for σ can be written as

$$\sigma = \frac{\tan\mu_2 - \tan(\beta-\theta)}{\tan\mu_2 + \tan(\beta-\theta)}. \tag{3.17a}$$

In this form, σ is readily computed from the shock jump conditions; $\tan(\beta-\theta)$ is obtained directly from (3.2). σ as a function of θ and M_1 is plotted in figure 2. Given an order of magnitude estimate for Φ, this figure provides a simple guide for the range of validity and the error introduced in Whitham's approximation. The lowest-order result for weak oblique shocks $\sigma = (g_1/2)\,\theta$ is also shown in the figure for comparison. One notes that, for the smaller values of θ, σ is a minimum at $M_1 = 2$. This value of M_1 corresponds to the minimum of the Mach number function g_1 in the first-order approximation for σ. For $M_1 < 3$, the linear approximation for σ is a reasonable description over much of the range of θ of interest, except for the region near $M_2 = 1$. For all M_1, σ approaches unity asymptotically with θ as the velocity behind the shock approaches sonic value, since $\tan\mu_2$ in (3.17*a*) approaches infinity asymptotically as M_2 approaches unity. Thus, if $\Phi \ll 1$, condition (3.26) is satisfied, and hence Whitham's approximation is valid for any oblique shock with supersonic flow behind it. When $\Phi = O(1)$, the limits of validity of criterion (3.26) are much more restricted, particularly at high Mach numbers, where σ is proportional to the hypersonic similarity parameter $M_1 \theta$.

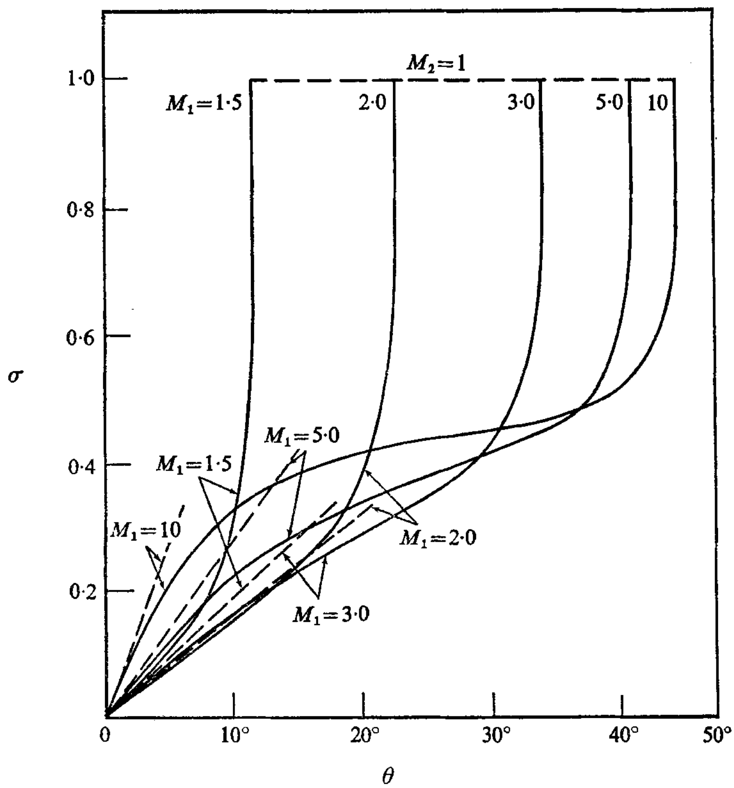

FIGURE 2. σ in condition (3.26) for the validity of Whitham's approximate characteristic rule. ——, exact result (3.17*a*); — — —, linear theory $\sigma = \frac{1}{2} g_1 \theta$.

(iii) *Compatibility conditions at the rear of the shock*

When a disturbance passes through an oblique shock wave, the change in shock strength must be consistent with the strength of the waves which propagate the disturbance downstream. Since each streamline that passes through a curved shock is a slip line, downstream compatibility conditions on pressure and flow deflection angle must be satisfied along the entire length of the shock. From (3.5), the variation in pressure at the rear of the shock, dp_2, comes from three sources: a Mach number gradient dM_1, an incoming pressure disturbance dp_1 and/or a change in the local turning angle of the flow $d\theta$. From (3.22) and (3.23) the pressure gradients so generated cannot be supported by the inviscid flow at the rear of the shock. This in turn produces waves of just sufficient strength to satisfy the slip line boundary condition. However, in this process, an unavoidable coupling between the downstream flow and the shock develops. The change in the local flow deflection at the rear of the shock, $d\theta_2$, which results from the variation dp_2, is determined, not by the shock relations, but by the flow equations downstream, i.e. (3.22). It is for this reason that the local turning angle of the flow, θ, cannot be determined independently of the downstream flow conditions.

Now, dp_2 in (3.5) must equal dp_2 in (3.23). This compatibility condition determines the basic equation for the local turning angle, θ:

$$-d\theta = \frac{1}{h+ap_1}(h\,d\theta_1 + b\,dp_1 + acp_1\,dM_1). \tag{3.28}$$

Equation (3.28) is result number (5) § 2. This is the important result of the derivation of the equations, in that the general shock refraction problem for $M_2 > 1$ has been reduced to the solution of a single differential relation, in which all wave interference effects downstream of the shock have been combined into a single parameter Φ. Under certain conditions, when (3.25) or (3.26) are satisfied, these effects can be ignored, and (3.28) solved directly, since the unknown parameter Φ is not required except to calculate higher-order corrections. For flows in which downstream wave interactions cannot be neglected, (3.28) is still readily solved, if the distribution of Φ along the rear of the shock is prescribed. Therefore, (3.28) also affords a simple approximate means of solving shock refraction problems, in which $\sigma\Phi$ is not small, which avoids the detailed solution of the governing system of partial differential equations. In problems of this type, one prescribes approximate distributions of Φ in much the same spirit that approximate distributions of velocity are chosen in Oseen type linearizations of the Navier–Stokes equations. This approximate approach to shock refraction problems is being studied further for flows where the incident waves behind the shock are important.

For large turning angles, but with $M_2 > 1$, (3.28) must be integrated numerically along the shock front. β is then determined from (3.2) or (3.6), p_2 from (3.1), and M_2 from the oblique shock relation,

$$M_2 = \frac{1}{\sin(\beta-\theta)}\left[\frac{1+\frac{1}{2}(\gamma-1)M_1^2\sin^2\beta}{\gamma M_1^2\sin^2\beta - \frac{1}{2}(\gamma-1)}\right]^{\frac{1}{2}}. \tag{3.29}$$

For small shock turning angles θ, (3.28) reduces to

$$-2\,d\theta = d\theta_1(1+\tfrac{1}{2}g_1\theta\Phi + O(\theta^2)) + \frac{dp_1}{a_1p_1}\left(1+\left(a_1-\frac{a_2}{a_1}\right)\theta - \frac{g_1}{2}\theta\Phi + O(\theta^2)\right)$$
$$+\,dM_1(c_2\theta + c_3\theta^2 - \tfrac{1}{2}c_2g_1\theta^2\Phi + O(\theta^3)), \tag{3.30}$$

while (3.29) becomes
$$M_2 = M_1 + f_1\theta + f_2\theta^2 + O(\theta^3), \tag{3.31}$$

$$f_1 = -\frac{M_1(1+\frac{1}{2}(\gamma-1)M_1^2)}{\omega_1},$$

$$f_2 = -f_1\frac{[(\gamma-1)M_1^4 - \frac{3}{2}(\gamma-1)M_1^2 - 1]}{2\omega_1}.$$

Note that there is no term $O(1)$ in the coefficient of dM_1 in (3.30). It is for this reason that the first higher-order correction for the pure shear case requires the retention of the c_3 term in (3.8). In § 5 and § 6, numerical and analytical solutions to (3.28) and (3.30) are presented, for various types of incoming disturbances where the criterion (3.25) or (3.26) is satisfied.

4. Upstream flow and initial conditions

The boundary conditions for the basic equation (3.28) or (3.30) are the values of M_1, p_1 and θ_1 just upstream of the shock. The initial conditions are the values of M_1, p_1, θ_1 and θ at the reference point where the integration is started; the subscript $(\ )_i$ will be used to denote this initial value. In the most general case, when M_1, p_1 and θ_1 depend explicitly on the spatial co-ordinates, one needs to integrate the equation of the shock front,

$$\frac{dY}{dx} = \tan(\beta+\theta_1), \tag{4.1}$$

in conjunction with (3.28) or (3.30). The shock angle β in (4.1) must be determined at each point along the shock from (3.6) or (3.11), as the case may be, since the position of the shock is one of the unknowns of the problem.

$d\theta_1$, dp_1, and dM_1 represent total differentials along the shock front in (3.28) and (3.30). Thus, when criterion (3.25) or (3.26) is satisfied, and terms involving Φ can be neglected, or if Φ is prescribed, the basic equation for $d\theta$ reduces to an ordinary differential equation. In addition, if (*a*) any two of the three differentials $d\theta_1$, dp_1, and dM_1, are zero, or (*b*) if p_1, θ_1, and M_1 can be related to one another and $\Phi = 0$ or is a function of the flow variables alone, then the solutions to (3.28) and (3.30) have some generality, since they are functions of the flow variables themselves, and do not depend explicitly on the spatial co-ordinates. The three flows of category (*a*) are: (i) pure shear, $dp_1 = d\theta_1 = 0$, $dM_1 \neq 0$; (ii) pure divergence, $dM_1 = dp_1 = 0$, $d\theta_1 \neq 0$; (iii) pure pressure disturbances, $dM_1 = d\theta_1 = 0$, $dp_1 \neq 0$. Flow (i) provides results of interest for the propagation of oblique shocks through boundary layers and free shear layers and shock refraction in a wind stratified atmosphere. Flow (ii) provides results of interest for the propagation of an oblique shock through a converging or diverging nozzle flow.

Three flows where the upstream conditions allow the determination of a relationship between the upstream flow quantities p_1, θ_1, and M_1, category (*b*), will be treated. These include the propagation of an oblique shock through: (iv) simple waves of the same family, (v) simple waves of the opposite family, (vi) irrotational non-simple wave regions where Φ_1, the ratio of the pressure gradients along the ξ and η characteristic directions in the upstream flow, is held constant. Flow (iv) is of interest, since it permits a simple check of the present analysis with higher-order thin airfoil theory results for the trailing edge shock wave. Flow (vi) is a hypothetical flow, in which Φ_1 will be allowed to assume a number of values between plus and minus infinity. The object is to obtain some insight into the more difficult problem of shock propagation in non-simple wave regions. The last flow considered, which does not belong to either category (*a*) or (*b*), is (vii) the propagation of a shock through a weak constant pressure rotational disturbance. The purpose will be to examine the relative contributions of small horizontal and vertical fluctuations in velocity to the sound waves generated when a shock propagates through a weak vortex. Flows (i)–(vii) provide basic insights into a large number of two-dimensional inhomogeneous free stream-shock interactions of interest for $M_2 > 1$. While some of the flows have the direct physical applications noted above, others represent highly idealized flow configurations. The primary objective in the selection of the seven cases is to gain understanding of the behaviour of the basic elements that enter into complex shock refraction problems when several elements are present.

5. Seven analytic solutions to the shock refraction relation

Assuming that criterion (3.25) is satisfied (i.e. that the strength of the incident waves at the rear of the shock is weak compared to that of the emitted waves which propagate the disturbance downstream), the basic refraction relation (3.30) simplifies to

$$-2\,d\theta = d\theta_1(1+O(\theta\Phi,\theta^2))+\frac{dp_1}{a_1p_1}\left(1+\left(a_1-\frac{a_2}{a_1}\right)\theta+O(\theta\Phi,\theta^2)\right)$$
$$+dM_1(c_2\theta+c_3\theta^2+O(\theta^2\Phi,\theta^3)). \quad (5.1)$$

Here terms of $O(\theta\Phi)$ and $O(\theta^2\Phi)$ have been neglected in comparison with terms of $O(\theta)$ and $O(\theta^2)$, respectively. The analytic solutions to (5.1) for the seven basic flow fields listed in §4 will now be presented. The importance of these results and their comparison with the numerical solution of (3.28) is discussed in §6. The solutions for β, p_2 and M_2 can be obtained directly from (3.11), (3.13), and (3.31) once θ is determined.

(i) *Pure shear*

$dp_1 = d\theta_1 = 0$, $dM_1 \neq 0$; and (5.1) reduces to

$$-2\,d\theta = (c_2\theta+c_3\theta^2+O(\theta^2\Phi,\theta^3))\,dM_1. \quad (5.2)$$

After substituting the expressions for the coefficients from (3.8), and rearranging terms, one obtains

$$\frac{d\theta}{dM_1}-\frac{M_1^2-2}{2M_1(M_1^2-1)}\theta = \frac{2}{M_1\omega_1}\left(1-\frac{\gamma+1}{4}\frac{M_1^4}{\omega_1^4}\right)\theta^2, \quad (5.3)$$

which is recognized as a Bernoulli equation, if the c_3 term is retained. The lowest-order non-trivial solution of (5.3) is obtained by dropping the term of $O(\theta^2)$ (i.e the right-hand side of (5.3)). The integral of the resulting homogeneous differential equation that satisfies the initial condition $\theta = \theta_2$ when $M_1 = M_{1i}$ is

$$\frac{\theta}{\theta_i} = \frac{M_{1i}}{M_1}\left(\frac{M_1^2-1}{M_{1i}^2-1}\right)^{\frac{1}{4}}. \quad (5.4)$$

The solution to the non-homogeneous equation (5.3), which retains terms of $O(\theta^2)$ and satisfies this initial condition, is

$$\frac{\theta}{\theta_i} = \frac{\dfrac{M_{1i}}{M_1}\left(\dfrac{M_1^2-1}{M_{1i}^2-1}\right)^{\frac{1}{4}}}{1+\dfrac{M_{1i}\theta_i}{(M_{1i}^2-1)^{\frac{1}{4}}}(F(M_1)-F(M_{1i}))}, \quad (5.5)$$

where $$F(M_1)-F(M_{1i}) = \int_{M_{1i}}^{M_1}\frac{1}{M^2(M^2-1)^{\frac{1}{4}}}\left(1-\frac{\gamma+1}{4}\frac{M^4}{(M^2-1)^2}\right)dM.$$

Both (5.4) and (5.5) give the local flow turning angle along the shock in terms of the local Mach number just upstream of the shock wave at any point in the shear layer, and its initial value M_{1i}, just outside the disturbed region.

(ii) *Pure divergence*

$dp_1 = dM_1 = 0$, $d\theta_1 \neq 0$; and (5.1) reduces to

$$-2d\theta = (1+O(\theta\Phi, \theta^2))\,d\theta_1. \tag{5.6}$$

The integral of (5.6) along the shock that satisfies the initial condition $\theta = \theta_i$ when $\theta_1 = 0$ (x-axis taken tangent to incoming streamline at initial station) is

$$\theta = \theta_i - \tfrac{1}{2}\theta_1. \tag{5.7}$$

Since no term of $O(\theta)$ appears in the coefficient of $d\theta_1$ in (5.6), this result, (5.7), is valid to $O(\theta\Phi)$ or $O(\theta^2)$, whichever is the larger. To this order, (5.7) states that the incident shock can be reduced to zero strength, or nearly doubled (see (3.13)), by turning the upstream flow through an angle plus or minus $2\theta_i$. Result (5.7) does not apply if θ_1 is increased beyond $2\theta_i$, since the Rankine–Hugoniot oblique shock relations, from which (3.5) and (3.28) or (3.30) are derived, no longer apply. Negative values of θ would imply an expansion fan rather than a shock wave. The interesting point is that the incoming disturbance is half transmitted along the shock wave and half transmitted along the downward running wave system at the rear of the shock. Thus, the shock does not rotate to lowest order with the turning of the incident stream.

(iii) *Pure pressure disturbance*

$d\theta_1 = dM_1 = 0$, $dp_1 \neq 0$; and (5.1) reduces to

$$-2d\theta = \left(\frac{1}{a_1} + \left(1 - \frac{a_2}{a_1}\right)\theta + O(\theta\Phi, \theta^2)\right)\frac{dp_1}{p_1}. \tag{5.8}$$

Since M_1 is a constant, a_1 and a_2 here are constants. If only the term of $O(1)$ in the coefficient of dp_1 is retained in (5.8), then the solution satisfying the initial condition $\theta = \theta_i$, when $p_1 = p_{1i}$, is

$$\theta - \theta_i = -\frac{(M_1^2-1)^{\frac{1}{2}}}{2\gamma M_1^2}\ln\left(\frac{p_1}{p_{1i}}\right). \tag{5.9}$$

One can also show from (3.12) and (5.8) that, to this order, $dp_2/p_2 = \frac{1}{2}dp_1/p_1$. Thus,

$$\frac{p_2}{p_{2i}} = \left(\frac{p_1}{p_{1i}}\right)^{\frac{1}{2}}. \tag{5.10}$$

If the term of $O(\theta)$ in the coefficient of dp_1 in (5.8) is retained, then the integral of (5.8) obeying the above initial condition is

$$\theta = -\frac{1}{2a_1\alpha}\left[1 + \left(\frac{p_1}{p_{1i}}\right)^{\alpha}\right] + \theta_i\left(\frac{p_1}{p_{1i}}\right)^{\alpha}, \tag{5.11}$$

where
$$\alpha = \frac{(\gamma-1)\,M_1^4 - 2(M_1^2-2)}{4\gamma M_1^2(M_1^2-1)}.$$

(iv) *Simple waves of the same family*

For simple wave regions no characteristic dimension is needed to characterize the upstream flow. The flow equations depend on the geometry only implicitly through the dependent variables. From (3.14), dp_1 and $d\theta_1$ are related along the ξ characteristic in the upstream flow (see figure 3) by

$$dp_1 - a_1 p_1 d\theta_1 = 0. \tag{5.12}$$

Equation (5.12) applies all along the upstream side of the shock, since conditions are constant along each wave of the η family. dM_1 and $d\theta_1$ are related through the Prandtl–Meyer relation for isentropic turning,

$$d\theta_1 = \frac{1}{f_1} dM_1, \tag{5.13}$$

where f_1 is given by (3.31). For simple waves of the same family, one does not have to impose criterion (3.25) to simplify (3.30) to $O(\theta^2)$, since the $\theta\Phi$ terms in the coefficients of this equation automatically cancel, when dp_1 and dM_1 are eliminated in favour of $d\theta_1$.

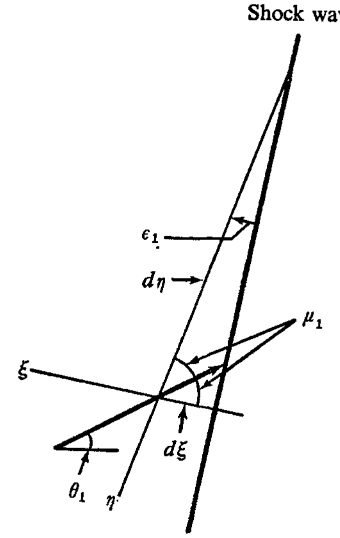

FIGURE 3. Characteristic co-ordinates ahead of shock wave.

Substituting (5.12) and (5.13) in (5.1), one obtains

$$d\theta = -d\theta_1 \left[1 + \left(a_1 - \frac{a_2}{a_1} + c_2 f_1\right)\frac{\theta}{2} + O(\theta^2)\right]. \tag{5.14}$$

However, one can show that the coefficient of the θ term is equal to zero for all M_1:

$$a_1 - \frac{a_2}{a_1} + c_2 f_1 = 0. \tag{5.15}$$

Thus, to $O(\theta^2)$, $d\theta = -d\theta_1$ or $d\theta_2 = 0$. Therefore,

$$\theta_2 = \theta_{2i}, \tag{5.16}$$

and the flow angle at the rear of the shock is unchanged to this order. Also, $\theta - \theta_i = \theta_{1i} - \theta_1$, which indicates that the shock is weakened by an amount equal to the cumulative strength of the Mach waves that intersect it, and will have decayed to zero strength at infinity, if $\theta_i = \theta_1 - \theta_{1i}$. This corroborates the well-known result for the behaviour of the trailing edge shock in higher-order thin airfoil theory (Lighthill 1954). In the latter theory, the flow conditions ahead of the shock are not described by the exact simple wave relations (5.12) and (5.13), as in the present theory, but by the second-order expression for the power series expansion of the Prandtl–Meyer relation about a uniform stream M_∞. To $O(\theta^3)$ the entropy reflections at the rear of the shock become important, and (5.16) is no longer valid.

(v) *Simple waves of the opposite family*

When a shock wave propagates through a Prandtl–Meyer expansion of the opposite family, one obtains, in place of (5.12) and (5.13),

$$dp_1 + a_1 p_1 d\theta_1 = 0, \tag{5.17}$$

$$d\theta_1 = -\frac{1}{f_1} dM_1. \tag{5.18}$$

For this case, the $\theta\Phi$ terms in the coefficients of $d\theta_1$ and dp_1 in (3.30) do not vanish, and criterion (3.25) is not satisfied automatically to $O(\theta^2)$. Equation (5.1) reduces to

$$2\,d\theta = \left[\left(a_1 - \frac{a_2}{a_1} + c_2 f_1\right)\theta + O(\theta\Phi, \theta^2)\right] d\theta_1. \tag{5.19}$$

The coefficient of the θ term is zero by virtue of (5.15) for all M_1. Therefore, to $O(\theta\Phi, \theta^2)$, $d\theta = 0$,

$$\theta = \theta_i, \tag{5.20}$$

and the local turning angle of the flow at the shock remains unchanged along its length. The strength of the shock, however, does vary, since the coefficients a_1 and a_2 in (3.13) are functions of M_1, which changes in accordance with the Prandtl–Meyer relation (5.18) for the upstream flow.

(vi) *Irrotational flow, non-simple waves*

In cases (iv), simple waves of the same family, and (v), simple waves of the opposite family, we examined the two limiting cases of isentropic, upstream flow: first only waves of the same family as the shock wave, then only waves of the opposite family to the shock wave. If Φ_1 is defined by the value of the ratio p_ξ/p_η just upstream of the shock, then $\Phi_1 = \pm\infty$ or $\Phi_1 = 0$ for simple waves of the same or opposite family as the shock respectively. In either case, $\theta\Phi$ is small compared to unity downstream of the shock. We would now like to consider more complicated upstream disturbances, in which waves of both families are present.

Provided the upstream disturbance is still isentropic, dp_1 and dM_1 are related by

$$dp_1 = \frac{a_1 p_1}{f_1} dM_1. \tag{5.21}$$

Substituting this result in (5.1), making use of (5.15), and neglecting terms of $O(\theta\Phi)$ and $O(\theta^2)$ in the coefficients of the differentials, one obtains

$$2d\theta = -d\theta_1 - \frac{dp_1}{a_1 p_1}. \tag{5.22}$$

dp_1 and $d\theta_1$ at the front of the shock in (5.22) can be related in much the same manner as were dp_2 and $d\theta_2$ at the rear of the shock in § 3 (ii). One first derives the equation of the shock front in terms of the ξ, η characteristic co-ordinates upstream of the shock wave, then derives the relation between dp_1 and $d\theta_1$ along this line, using the p, ϕ characteristic relations (3.14).

From figure 3, one obtains in place of (3.17),

$$d\xi = \zeta d\eta, \tag{5.23}$$

where

$$\zeta = \frac{\sin \epsilon_1}{\sin (2\mu_1 + \epsilon_1)} = \frac{\tan \beta - \tan \mu_1}{\tan \beta + \tan \mu_1}.$$

Following an analogous procedure to that used in the derivation of (3.22), one now finds that, along the upstream side of the shock wave,

$$dp_1 + a_1 p_1 \left(\frac{1+\zeta\Phi_1}{1-\zeta\Phi_1}\right) d\theta_1 = 0 \quad \text{on} \quad Y(x). \tag{5.24}$$

ζ in (5.24), like σ in (3.22), can be expressed as a power series in θ:

$$\zeta = \tfrac{1}{2} g_1 \theta + O(\theta^2), \tag{5.25}$$

which to $O(\theta)$ is the same as that for σ. Combining (5.24) and (5.25), substituting for $d\theta_1$ in (5.22), and then eliminating dp_1 in favour of dM_1 from (5.21), one finds, after rearranging terms,

$$\frac{d\theta}{dM_1} + \frac{g_1 \Phi_1 \theta}{f_1(2 + g_1 \Phi_1 \theta)} = 0, \tag{5.26}$$

where the coefficients of the Φ_1 terms are correct to $O(\theta)$. Unlike the flow behind the shock, $\Phi_1 \theta$ need not be small compared to unity, since we shall want to investigate flows for which Φ_1 can take on any value in the range $-\infty < \Phi_1 < \infty$. Thus, (5.26) has a singularity at

$$\theta = -\frac{2}{g_1 \Phi_1}, \tag{5.27}$$

at which point $d\theta/dM_1$ is infinite. The second term in (5.26) changes sign as θ passes through this value, so that a different behaviour can be expected on each side of the singularity.

Equation (5.26) does not have a closed form solution for arbitrary values of Φ_1 over the entire range. However, if $|g_1 \Phi_1 \theta| \ll 2$, it simplifies, after inserting the expressions for f_1 and g_1, to

$$\frac{d\theta}{dM_1} - \left(\frac{\gamma+1}{8}\right) \frac{M_1^3}{(M_1^2 - 1)\left(1 + \frac{\gamma+1}{2} M_1^2\right)} \Phi_1 \theta = 0. \tag{5.28}$$

Equation (5.28) is an exact differential if Φ_1 is a constant (i.e if the relative strengths of the incoming waves of both families upstream of the shock is constant along its length). The solution that satisfies the initial condition $\theta = \theta_i$ at $M_1 = M_{1i}$ is

$$\frac{\theta}{\theta_i} = \left(\frac{G(M_1)}{G(M_{1i})}\right)^{-\Phi_1}, \tag{5.29}$$

where $$G(M) = (M^2-1)^{-\frac{1}{8}}\,(1+\tfrac{1}{2}(\gamma-1)\,M^2)^{-1/\{4(\gamma-1)\}}.$$

This solution reduces to (5.20) when $\Phi_1 = 0$, simple waves of the opposite family.

When $|g_1 \Phi_1 \theta|$ is not $\ll 2$, (3.28) must be integrated numerically. First, dp_1 and $d\theta_1$ are eliminated in favour of dM_1, through (5.21) and (5.24). Equation (3.28) becomes

$$\frac{d\theta}{dM_1} = -\frac{1}{h+ap_1}\left[-\frac{h}{f_1}\left(\frac{1-\zeta\Phi_1}{1+\zeta\Phi_1}\right)+\frac{a_1 p_1 b}{f_1}+acp_1\right]. \tag{5.30}$$

The right-hand side of (5.30) is only a function of M_1, θ and Φ_1 when h is given by (3.27). Therefore, (5.30) can be readily integrated for any fixed value of Φ_1. For simple waves, (5.30) simplifies to

$$\frac{d\theta}{dM_1} = -\frac{1}{h+ap_1}\left[\pm\frac{h}{f_1}+\frac{a_1 p_1 b}{f_1}+acp_1\right], \tag{5.31}$$

where the minus and plus signs apply to the same ($\Phi_1 = \pm\infty$) and opposite ($\Phi_1 = 0$) families, respectively. Numerical solutions of (5.30) and (5.31) are presented in §6.

(vii) *Small constant pressure rotational disturbances*

In this final case, the pressure is constant, $dp_1 = 0$, and (5.1) becomes

$$-2\,d\theta = d\theta_1(1+O(\theta\Phi,\theta^2))+dM_1(c_2\theta+c_3\theta^2+O(\theta^2\Phi,\theta^3)). \tag{5.32}$$

If we neglect terms of $O(\theta)$ and $O(\theta\Phi)$, (5.32) reduces to

$$2\,d\theta = -d\theta_1. \tag{5.33}$$

The solution, which satisfies the initial condition $\theta = \theta_i$ when $\theta_1 = 0$, is

$$\theta = \theta_i - \tfrac{1}{2}\theta_1. \tag{5.34}$$

This is the same result as case (ii), pure divergence, (5.7). Thus, if the Mach number fluctuations normal and parallel to the streamwise direction are of the same order, the effects of the fluctuations in magnitude of the mainstream velocity are of higher order than the fluctuations in direction, since the terms involving dM_1 do not appear in (5.33). Thus, the dominant contribution in this case comes from the small angular fluctuations of the oncoming mainstream. The downstream disturbed pressure field produced by a shock-vortex interaction arises principally from a focusing and defocusing of the shock front due to the local convergence and divergence of incoming stream tubes, variations in temperature and horizontal velocity component being a higher-order effect.

It is interesting to observe how the ordering of the upsteam disturbances is altered when the constant pressure condition $dp_1 = 0$, used in obtaining (5.32), is relaxed. For a completely general supersonic upstream flow, dp_1 and $d\theta_1$ are related by (5.24) along the upstream side of the shock. Substituting this result in (5.1), neglecting terms of $O(\theta^2)$, and retaining downstream wave interaction effects of $O(\theta\Phi)$, for greater generality, one obtains

$$d\theta = -\frac{d\theta_1}{2}\left[1+\frac{g_1}{2}\theta\Phi-\left(\frac{1+\zeta\Phi_1}{1-\zeta\Phi_1}\right)\left(1+\left(a_1-\frac{a_2}{a_1}\right)\theta-\frac{g_1}{2}\theta\Phi\right)\right]-\frac{dM_1}{2}c_2\theta. \tag{5.35}$$

Only when Φ_1 approaches zero, and the upstream flow is dominated by waves of the opposite family to the shock, will the coefficients of the $d\theta_1$ and dM_1 terms in (5.35) be of the same order. Therefore, result (5.34) will in general apply provided Φ_1 is not $\leqslant O(\theta)$ and $d\theta_1 \neq 0$.

This concludes the presentation of the seven analytic solutions to the shock refraction relation (5.1). In § 6 we present in graphical form analytical and numerical results for each case, and discuss the important points.

6. Discussion of results and comparison of analytical and numerical solutions

In § 6 we discuss in greater detail the analytic solutions of § 5 and compare them with numerical solutions of (3.28), the basic equation from which all the others were derived. In the numerical solutions, we shall assume that the criterion (3.26) is satisfied, and that the wave interactions behind the shock can be neglected; h is then given by (3.27), and (3.28) reduces to the same relation that would obtain from a direct application of Whitham's approximate characteristic rule.

(i) *Pure shear*: $d\theta_1 = dp_1 = 0$; $dM_1 \neq 0$

Figures 4 and 5 present the results for case (i). This is a comparison of the numerical solution of (3.28) with the analytic solution of (5.4) and (5.5), the second- and third-order equations respectively. The problem of the propagation of a shock through a steady supersonic two-dimensional shear flow was first studied by Moeckel (1952). His basic equation is a special case of (3.28), in which $dp_1 = d\theta_1 = 0$ and criterion (3.26) is satisfied. Thus, the numerical solutions in figure 4 correspond to the Moeckel–Whitham approximate theory; the analytical solutions are new. In figures 4 and 5, θ_i is taken as the turning angle of the flow at the shock at the point where M_2 is equal to one (in other words, the Mach number behind the shock is just sonic). Therefore, this curve corresponds to the case of a shock propagating outward through a shear layer from the point where M_2 is equal to one into higher Mach number regions, or to the case of a shock propagating into a shear layer from an external flow toward a point on the sonic line. The turning angle θ for the analytic solution was matched with the numerical solution at the point where M_1 was equal to 10. If one interpolates between the curves in figure 4, θ can be determined as a function of M_1 for any initial turning angle which falls beneath the curve $\theta_i = 20$ degrees. For example, given a shock entering a shear layer where the external Mach number is 7, and the turning angle at incidence is 11°, θ is picked up through interpolation between the curves θ_i equal 18° and 20°. The interpolation continues as one proceeds into the shear layer. Thus, figures 4 and 5 are charts for the case of shock interaction with a pure shear layer, to be used in much the same manner as the oblique shock charts.

Notice that at hypersonic Mach numbers the turning angle θ can vary by a factor of two or more when a shock wave propagates through a shear layer such as the supersonic portion of a boundary layer. A related result obtains from an earlier theory for the behaviour of expansion fans propagating through supersonic boundary layers (Weinbaum 1966). In the latter work, it is shown that an

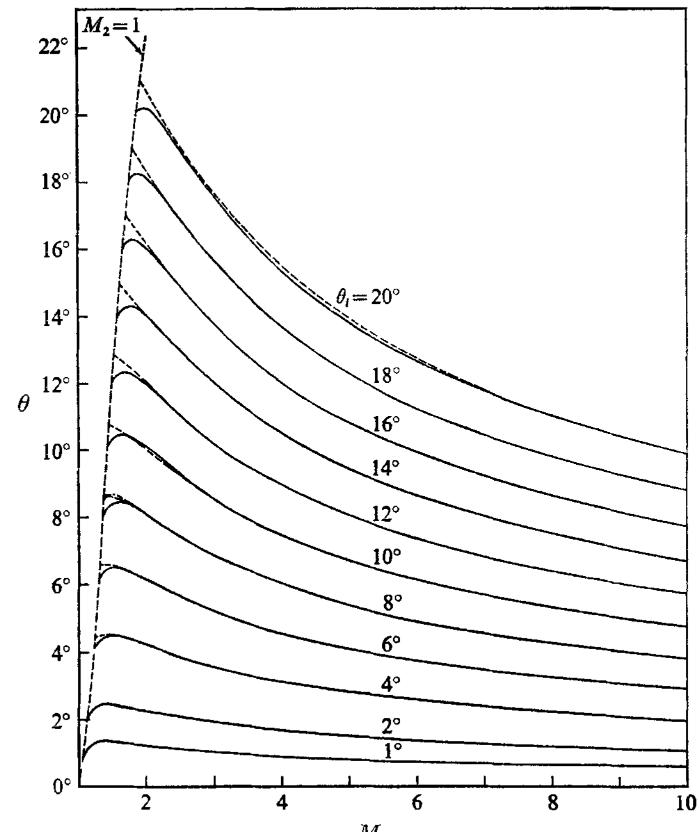

FIGURE 4. Comparison: ——, numerical result (Moeckel–Whitham approximation); - - - - - -, second-order result (5.4); –·–·–·–, third-order result (5.5), for the variation of local shock deflection angle in a shear layer. θ_l value of θ at $M_2 = 1$.

expansion fan, interacting with a shear layer, will generate large differences in turning angle and large gradients in pressure along the rear of the expansion fan at high Mach numbers. There is remarkable agreement (see figure 4) between the second- and third-order analytic theory and the numerical solution, even for local turning angles as large as 20°, except as one approaches $M_2 = 1$. To treat the region near $M_2 = 1$ properly, the analysis would have to be developed, not in the local turning angle, but in the transonic similarity variable. At the larger turning angles and higher Mach numbers, the numerical solutions may for the reasons already discussed become inaccurate. Φ for a hypersonic shear layer can be of $O(1)$. Thus, if the upward reflected waves behind the shock are to be neglected, σ in condition (3.26) must be small compared to unity. Figure 2 thus serves as a convenient guide for the accuracy of these numerical solutions.

Figure 5 is the plot of the variation in shock strength *versus* M_1 for case (i), pure shear. In this figure, the second-order theory corresponds to the Busemann expression for the pressure change across the shock, (3.13), and is based on solution (5.4) for θ. The third-order result was obtained from the third-order expression given for the pressure change in Lighthill (1954, §E, 3), and is based

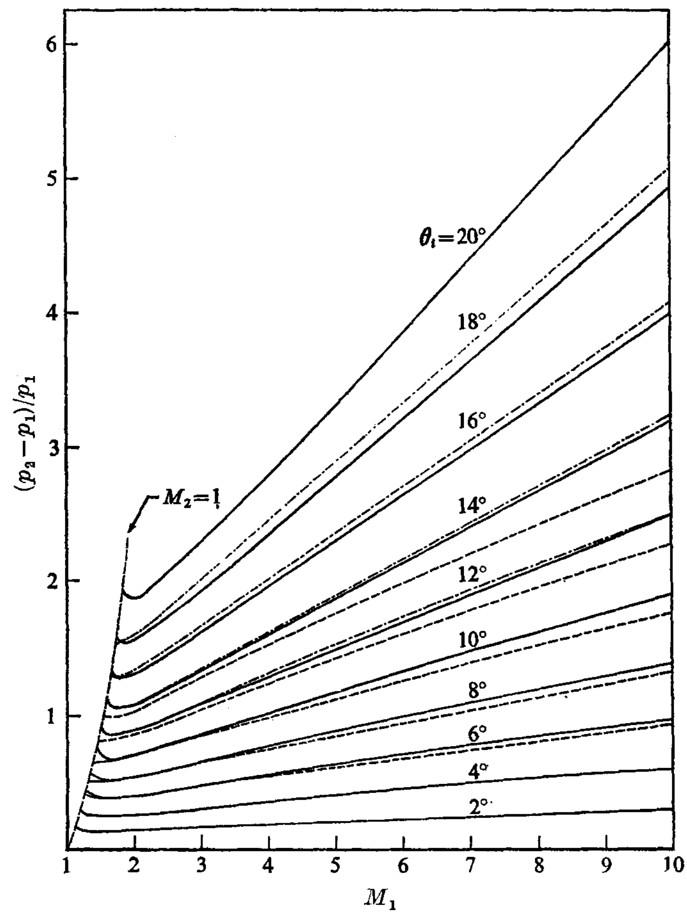

FIGURE 5. Variation of shock strength in a shear layer: ——, numerical (3.1); - - - - -, Buseman second-order; —·—·—·—, third-order Lighthill (1954). Local values of θ obtained from respective curves in figure 4.

on solution (5.5) for θ. The behaviour for the shock strength as a function of M_1 is just the inverse of the behaviour that was observed for the local turning angle of the shock (figure 4): θ decreases monotonically as a function of Mach number beyond the transonic region, whereas the shock strength increases with increasing Mach number. The agreement between the analytic solutions and the numerical solution for the strength of the shock, as based upon either the Busemann second-order or the third-order theory, is not nearly as good as that for the local turning angle θ. The authors believe the explanation for this is that the equation for θ is based, not on the absolute value of the pressure change that occurs across the shock, but on the pressure gradient dp_2 that is being generated at the rear of the shock. This pressure gradient, which gives rise to the emitted waves, depends on relative changes in shock strength (i.e the slope of the shock strength curves in figure 5), and not on the absolute values of the shock strength, the quantity of interest in the higher-order theory of Busemann and Lighthill. This argument is consistent with the observation that the numerical and analytic solution curves in figure 5 are very nearly parallel, even for the larger turning angles in the figure, except in the transonic region.

(ii) *Pure divergence*: $dp_1 = dM_1 = 0$; $d\theta_1 \neq 0$

Figure 6 presents the results for case (ii), pure divergence. The local turning angle θ is plotted as a function of the upstream flow direction θ_1, for various initial turning angles θ_i, for an incident stream at $M_1 = 3{\cdot}0$. The agreement between the numerical solution of (3.28) with h given by (3.27) and the second-order analytical solution (5.7) is striking. Small departures from the numerical

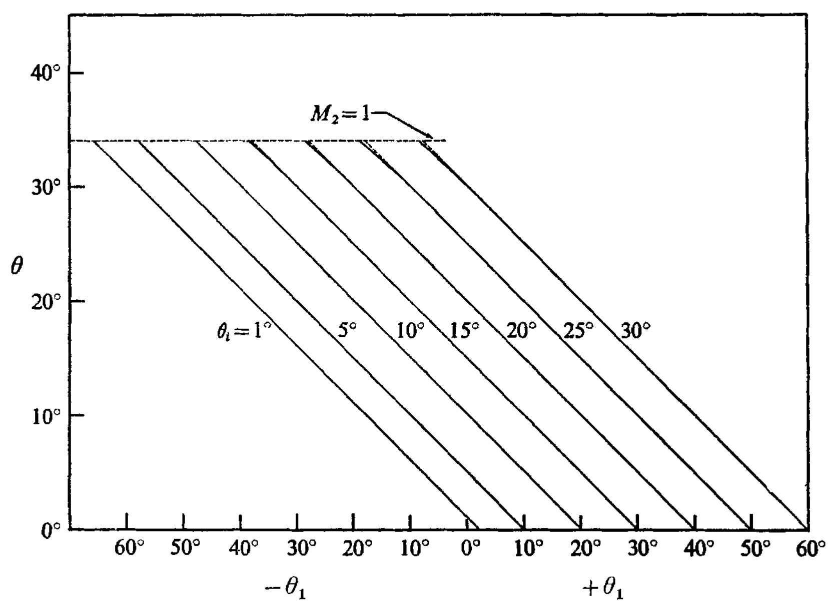

FIGURE 6. Comparison: ——, numerical solution ((3.28) with condition (3.26) imposed); - - - - -, second-order solution (5.7), for the variation of the local shock turning angle θ in a converging or diverging stream. $M_1 = 3$.

solution exist only at the very largest values of θ_1 near $M_2 = 1$, and then only for initial turning angles in excess of 20°. Excellent agreement was also observed for other values of M_1 (not shown), the difference between the numerical and second-order analytical solutions increasing slightly with increasing Mach number. At $M_1 = 10$, maximum deviations were still confined to less than 10 % over the entire range of θ_1, for which the theory is valid for $\theta_i = 30°$. According to the second-order theory (5.7), the shock strength is reduced to zero if the flow ahead of it is turned through a positive angle $2\theta_i$, or doubled if θ_1 is turned through an angle $-2\theta_i$. This is an important point. The cumulative turning of the flow by the waves emitted at the back of the shock can be comparable to the turning angle produced by the shock, if the angular deflections of the incident stream are of the same order as the turning angle of the shock in an undisturbed stream. This effect should then be an important consideration for very weak shocks propagating through a small disturbance field, e.g. the propagation through the earth's boundary layer of the far field N wave produced by an aircraft flying at high altitude.

No numerical solutions to case (iii), pure pressure disturbance, were obtained.

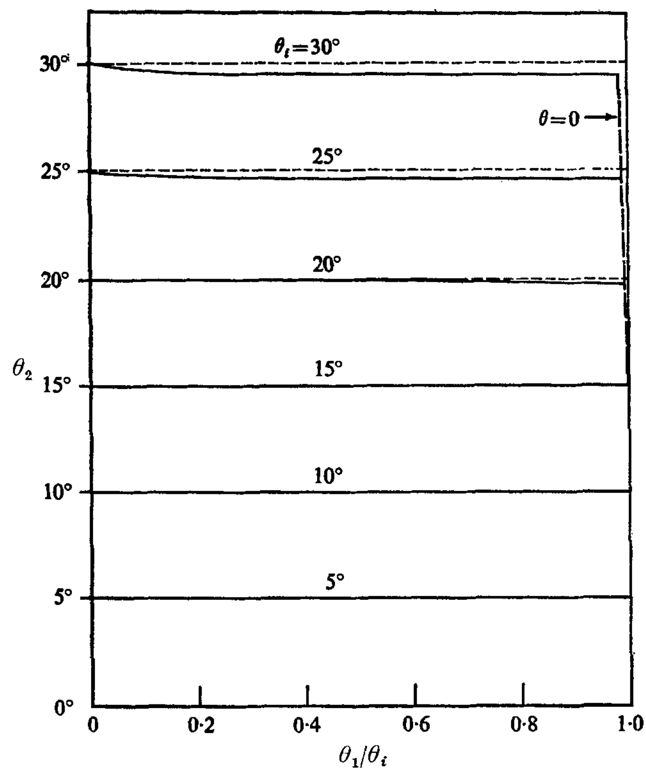

FIGURE 7. Comparison: ——, numerical result ((5.31) with condition (3.26) imposed); - - - - -, second-order result (5.16), for the propagation of an oblique shock into a Prandtl–Meyer expansion fan of the same family. $M_{1i} = 3$.

(iv) *Simple waves of the same family*

The cases calculated and shown in figure 7 are for various strength oblique shocks entering an expansion fan of the same family at an initial Mach number $M_{1i} = 3$. The numerical solution corresponds to (5.31), taken with the minus sign, with h given by (3.27). The shock is gradually attenuated by the Prandtl–Meyer expansion and eventually reduced to zero strength at infinity in space. The solutions presented herein are independent of the spatial co-ordinates, since we are looking at exact differentials involving just the flow variables. The turning angle at the back of the shock according to the second-order theory (5.16) remains unchanged, $\theta_2 = \theta_{2i}$. The waves entering the shock turn the upstream flow progressively upward, until the flow directions just upstream and downstream of the shock are parallel, and the shock is reduced to zero strength. For example, consider the case where the incident shock strength is $\theta_i = 15°$; if the flow ahead of it θ_1 is turned up 15°, then to $O(\theta^2)$ the shock is no longer required, since, to this order, the oblique shock relations are equivalent to a Prandtl–Meyer expansion of the same family. The line drawn through the end points of the numerical

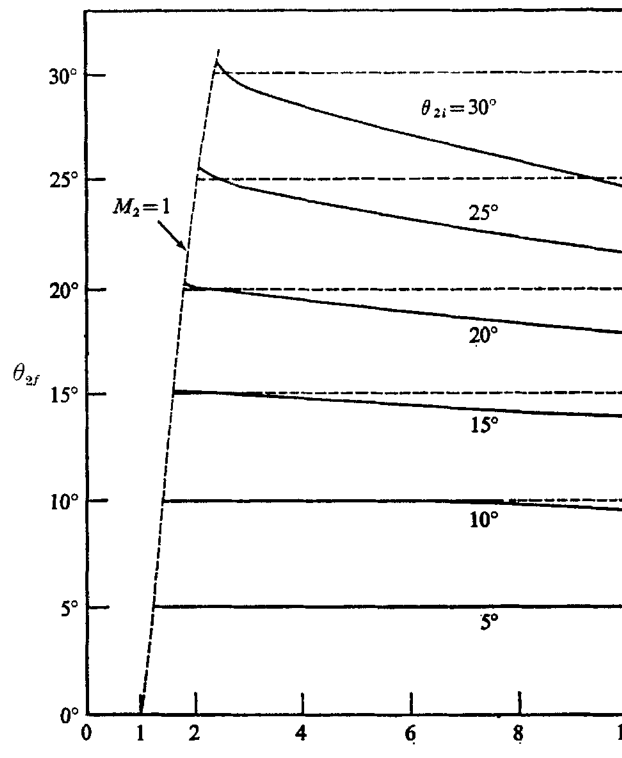

FIGURE 8. Prandtl–Meyer turning angle θ_{2f} required to completely attenuate an oblique shock of the same family as a function of incident Mach number M_{1i}. ——, numerical solution (5.31); - - - - -, second-order solution (5.16).

curves gives the values of θ_1/θ_i at which θ becomes equal to zero (i.e. where the shock is of zero strength). The departure of this value of θ_1/θ_i from unity, the value predicted by the second-order theory, which is isentropic, is a measure of the strength of entropy reflections or non-isentropic behaviour of the shock. These reflected waves, whose strength is $O(\theta^3)$, grow in importance as the shock strength is increased. It is interesting to note in figure 7 that, for an initial Mach number of three, this third-order effect produces less than 2 % discrepancies in the shock propagation results for turning angles as large as 30°. The numerical solutions, which neglect the wave interactions behind the shock, should be a very good approximation to the exact solution for this case. As noted earlier in the derivation of (5.14), downstream wave interference effects are of $O(\theta^2\Phi)$, rather than $O(\theta\Phi)$, for this flow configuration.

Figure 8 plots θ_{2f} against M_{1i} for simple waves of the same family. The second-order theory is again given by (5.16), and θ_{2f} refers to the final turning angle at the back of the shock when the shock has been reduced to zero strength at infinity. This graph shows the asymptotes of the curves in figure 7 at $\theta = 0$, as a function of different initial incident Mach numbers for the oblique shock. At higher Mach numbers, the discrepancy between the numerical and second-order solutions increases. This occurs, because, for a given shock turning angle, the shock strength increases as one proceeds to higher Mach numbers, and the entropy reflections neglected in the second-order theory become increasingly more important. This effect is shown clearly in figure 8.

(v) *Simple waves of the opposite family*

The numerical solutions shown in figure 9 for this case are obtained from (5.31) with the plus sign, where we again assume that condition (3.26) is satisfied. A

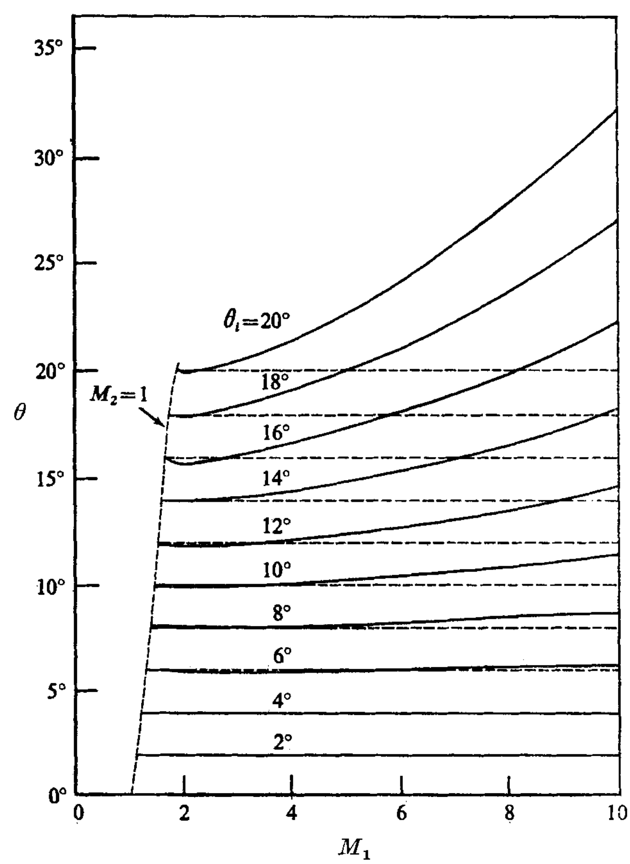

FIGURE 9. Comparison: ——, numerical result ((5.31) with condition (3.26) imposed); - - - - -, second-order result (5.20), for the propagation of an oblique shock through a Prandtl–Meyer expansion fan of the opposite family. θ_i value of θ at $M_2 = 1$.

numerical solution of the resulting equation, for the case $\theta_i = 30°$, and $M_{1i} = 3$ has been previously presented by Rosciszewski (1960), who compared his numerical solution with an exact characteristics calculation, and demonstrated very close agreement between the two solutions for this case. One would, therefore, expect that condition (3.26) is not seriously violated for the smaller turning angle cases shown in figure 9, and that the numerical solution curves are a good representation of the exact solution. The second-order analytic result (5.20) states that the turning angle at the shock θ is the same all along the shock $\theta = \theta_i$; thus the analytic theory in figure 9 shows as a horizontal line. When θ_i is small, there is close agreement between the second-order theory and the numerical solution at all values of M_1. The integration of (5.31) is started at a value M_{1i}, for which M_2 is just sonic. Thus, for weak shocks, the second-order theory appears valid even for large angle expansion fans. For shock turning angles $\theta_i > 10°$, there seems to be a significant discrepancy between the numerical and theoretical solutions, larger than one would expect bearing in mind the previous comparisons.

(vi) *Irrotational flow, non-simple waves*

This is the more complicated case, in which isentropic upstream disturbances of both families of waves are present. Thus, $\Phi_1 = p_\xi/p_\eta$ on the upstream side of the shock may take on values from plus to minus infinity. Figure 10 shows by comparison with figures 9 and 7, respectively, that the results, for the limiting cases $\Phi_1 = 0$ (simple waves of the opposite family) and $\Phi_1 = \pm\infty$ (simple waves of the same family), approach their correct limits. The analytic solution (5.29) for θ/θ_i is independent of θ_i, and thus appears as a single curve for each value of Φ_1 in figure 10. The accuracy of the analytic solution depends on the smallness of the quantity $|g_1 \Phi_1 \theta|$. One notes that the solution curves as Φ_1 approaches plus or minus infinity asymptote to the same limiting values. This occurs, because it does not matter whether p_η approaches zero from the plus or minus side. When $\Phi_1 = 1$, the waves of each family are both of the same character and same strength, so that θ_1 does not change in the streamwise direction in the upstream flow. When $\Phi_1 = -1$, the waves of each family are of the same strength, but of opposite character, so that θ_1 does not change in the direction normal to streamlines in the upstream flow.

(vii) *Small constant pressure rotational disturbances*

The important result for this case was already noted in conjunction with (5.32). The contribution to the perturbation pressure field downstream of the shock caused by the interaction between the shock and a small constant pressure upstream disturbance is to first order produced by the variation in angle of incidence of the streamlines ahead of the shock (result (5.7) or (5.34)), whereas the shear contribution is a second-order effect. It is interesting that this result is in contradistinction to the assumptions used in ordering the governing fluid equations for the propagation of sound waves through a medium with small unsteady irregularities (Lighthill 1953; Kraichnan 1953; Batchelor 1957; Crow 1969). In these papers, it is assumed that, in the interaction between an eddy and a sound wave, the Mach number fluctuations normal and parallel to the undisturbed incident stream direction generate pressure disturbances, which are of the same order of magnitude, and that the perturbation velocity field due to the passage of the sound wave is small compared to the velocity field of the

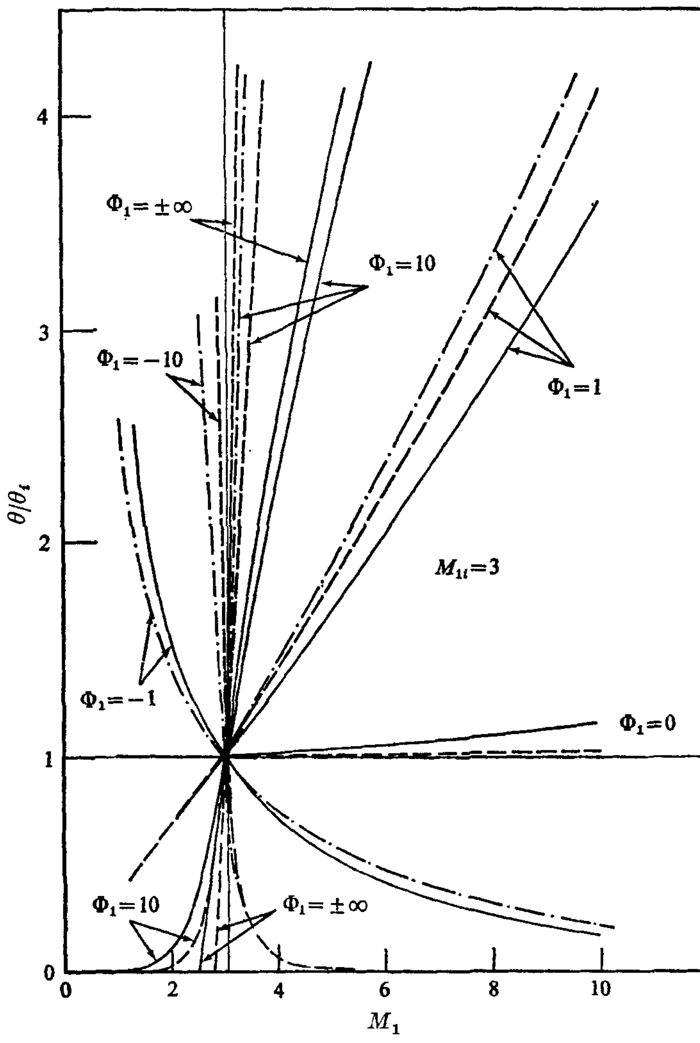

FIGURE 10. Comparison: – – – –, numerical result ((5.31) with condition (3.26) imposed), $\theta_i = 2°$; ——, $\theta_i = 10°$; — · —, analytical result (5.29), for the propagation of an oblique shock through an isentropic non-simple wave region with Φ_1, the value of the ratio of p_ξ/p_η on the upstream side of the shock constant. $M_{1i} = 3$.

turbulent fluctuations. The latter implies that the sound wave rotates to lowest order with the incident stream direction. The difference between a sound wave of wavelength λ generated by a vortex or vibrating boundary (where λ is large compared with a molecular diffusion length) and a weak shock wave, whose thickness is based on the latter dimension, is important to the derivation of acoustic scattering theory. A first-order or singly-scattered wave theory is based on a first-order Born approximation, which becomes strongly divergent toward very short sound wavelengths. Thus, both the local wave-vortex interaction process and the treatment of the governing fluid equations may be fundamentally different for sound waves and very weak shock waves.

7. Concluding remarks

The approach developed in the present study, wherein a single unknown interaction parameter Φ is introduced to describe the cumulative effect at the shock front of all the wave interactions that occur downstream of the shock, can obviously be applied to numerous other steady and unsteady shock propagation problems in non-uniform flow. Many of these problems are described in Whitham (1958) and Rosciszewski (1960), where the approach employed is based on the characteristic approximation. The present approach has the important advantage that it offers the possibility of obtaining improved results by specifying the distribution of the unknown interaction parameter Φ.

To illustrate the application of the present method to other problems, we shall briefly consider the case of one-dimensional unsteady shock propagation with no area changes. This case has been selected, because it is one of the illustrations Whitham (1958) chose to describe in detail. For normal shocks, the expansion variable for the shock jump relations would be the shock strength, instead of the shock turning angle θ, and (3.5) and (3.8) would be modified accordingly. For one-dimensional unsteady flow the characteristic relations,

$$dp \pm \rho a du = 0 \quad \text{on} \quad \frac{dt}{dx} = \frac{1}{u \pm a}, \tag{7.1}$$

would replace (3.14). If ξ, η denote the rightward and leftward running characteristic co-ordinates in the (t, x) diagram, then one can show from geometry, by analogy with the derivation of (3.17), that

$$d\xi = \sigma d\eta, \tag{7.2}$$

$$\sigma = \frac{\sin\left(\tan^{-1}\left(\frac{1}{U}\right) - \tan^{-1}\left(\frac{1}{u+a}\right)\right)}{\sin\left(\tan^{-1}\left(\frac{1}{U}\right) - \tan^{-1}\left(\frac{1}{u-a}\right)\right)},$$

where U is the shock velocity, and $\tan^{-1}(1/U)$ is the local slope of the shock trajectory in the (t, x) plane. Again, one introduces the unknown interaction parameter $\Phi = p_\xi/p_\eta$ and derives the expression relating dp and du along the rear of the shock trajectory. This derivation parallels that of (3.22), and one obtains

$$dp + \rho a\left(\frac{1+\sigma\Phi}{1-\sigma\Phi}\right) du = 0 \quad \text{on} \quad \frac{dt}{dx} = \frac{1}{U}, \tag{7.3}$$

which, when written in the form of (3.22a), becomes

$$dp + \rho a du = -\rho a\left(\frac{2\sigma\Phi}{1-\sigma\Phi}\right) du. \tag{7.4}$$

Now the Chisnell–Whitham approximate solution for one-dimensional unsteady flow depends on the smallness of the left-hand side of (7.4). In Whitham's formulation, $dp + \rho a\,du$ along the shock trajectory is the quantity,

$$\left(\frac{1}{U} - \frac{1}{u+a}\right)(p_t + \rho a u_t)\,dx \tag{7.5}$$

(Whitham 1958, (12)). For weak shocks, Whitham argues that the first factor in (7.5) is small, whereas for stronger shocks he requires that $p_t + \rho a u_t$ be small if the approximate rule is to be accurate. These arguments are equivalent to the condition that $\sigma\Phi \ll 1$ in (7.4). Equation (7.5) is the counterpart of (3.22*b*) for two-dimensional steady flow.

A preliminary version of this paper was presented at the AIAA 7th Aerospace Sciences Meeting, New York, 20–22 January 1969; it is available as AIAA Paper 69–39. We thank A. George, of Cornell University, for calling to our attention the similarity between the present theory and Whitham's rule. We also thank Drs J. Cole and S. Crow for the helpful discussions we had with them, at the Boeing Scientific Research Laboratories.

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